

On a class of transformations of sequences of complex numbers

Ilia D. Mishev *

Abstract

In this paper we consider a transformation L_a of sequences of complex numbers. We find the inverse transformation of L_a as well as the inverse of a related transformation \tilde{L}_a . We explore a connection to the binomial transform and significantly generalize a previously known result. We also obtain new relations among many classical hypergeometric orthogonal polynomials as well as new formulas for sums involving terminating hypergeometric series.

1 Introduction

Let ω denote the complex linear space of all sequences $x = (x_n)_{n=0}^{\infty}$ of complex numbers. Let $a_{n,k}$ be complex numbers, where n and k are integers with $n \geq 0, 0 \leq k \leq n$. We can define a linear transformation

$$L : \omega \rightarrow \omega$$

$$x = (x_n)_{n=0}^{\infty} \mapsto L(x) = (L(x)_n)_{n=0}^{\infty}$$

by

$$L(x)_n = \sum_{k=0}^n a_{n,k} x_k, \quad n \geq 0. \quad (1.1)$$

*Department of Mathematics, University of Colorado at Boulder, Campus Box 395, Boulder, CO 80309-0395, U.S.A. E-mail address: ilia.mishev@colorado.edu

One classical example of such a transformation is the binomial transform (introduced by Knuth in [3]) defined by

$$\hat{x}_n = \sum_{k=0}^n (-1)^k \binom{n}{k} x_k, \quad n \geq 0.$$

Given a transformation L of the kind defined in (1.1), L is invertible if and only if $a_{n,n} \neq 0$ for all $n \geq 0$. If L is invertible, its inverse transformation L^{-1} is of the form

$$L^{-1}(x)_n = \sum_{k=0}^n b_{n,k} x_k, \quad n \geq 0,$$

for some $b_{n,k} \in \mathbb{C}$ with $n \geq 0, 0 \leq k \leq n$. The inverse transformation can also be written as

$$x_n = \sum_{k=0}^n b_{n,k} L(x)_k, \quad n \geq 0.$$

As an example, the binomial transform is invertible and is equal to its own inverse (see [6]), which can be expressed by

$$x_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \hat{x}_k, \quad n \geq 0.$$

Remark 1.1. We note that if

$$y_n = \sum_{k=0}^n a_{n,k} x_k \Leftrightarrow x_n = \sum_{k=0}^n b_{n,k} y_k$$

is a pair of inverse transformations and $(c_n)_{n=0}^{\infty}$ is a sequence of nonzero complex numbers, then

$$y_n = \sum_{k=0}^n a_{n,k} c_k x_k \Leftrightarrow x_n = \frac{1}{c_n} \sum_{k=0}^n b_{n,k} y_k$$

is another pair of inverse transformations.

A large number of pairs of inverse transformations are examined by Riordan in [6].

In this paper, we will consider the following family of transformations:

Definition 1.2. Let $a \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$. We define the transformation

$$\begin{aligned} L_a : \omega &\rightarrow \omega \\ x = (x_n)_{n=0}^\infty &\mapsto L_a(x) = (L_a(x)_n)_{n=0}^\infty \end{aligned}$$

by

$$L_a(x)_n = \sum_{k=0}^n (-n)_k (n+a)_k x_k, \quad n \geq 0, \quad (1.2)$$

where, for $\gamma \in \mathbb{C}$ and k a nonnegative integer, the rising factorial $(\gamma)_k$ is given by

$$(\gamma)_k = \begin{cases} \gamma(\gamma+1) \cdots (\gamma+k-1), & k > 0, \\ 1, & k = 0. \end{cases}$$

We require that $a \notin \{-1, -2, -3, \dots\}$ since otherwise the transformation L_a is not invertible as in that case $(n+a)_n = 0$ for $1 \leq n \leq |a|$.

The transformation L_a arises naturally from the definitions of many classical hypergeometric orthogonal polynomials. It also comes up in the summation formulas of many terminating hypergeometric series.

Remark 1.3. From the inverse transformation of the binomial transform, Remark 1.1, and the formula

$$(-1)^k \binom{n}{k} = \frac{(-n)_k}{k!},$$

we obtain the pair of inverse transformations

$$y_n = \sum_{k=0}^n (-n)_k x_k \Leftrightarrow x_n = \frac{1}{n!} \sum_{k=0}^n \frac{(-n)_k}{k!} y_k.$$

The transformation L_a given in Definition 1.2 has the form

$$y_n = \sum_{k=0}^n (-n)_k (n+a)_k x_k, \quad a \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}.$$

A related transformation \tilde{L}_a (see Definition 3.2) that we will also study has the form

$$y_n = \sum_{k=0}^n \frac{(-n)_k}{(1+a+n)_k} x_k, \quad a \in \mathbb{C} \setminus \{-2, -3, -4, \dots\}.$$

In Section 3 we will find the inverse of the L_a transformation using a classical summation formula due to Dixon. We will also find the inverse of the related \tilde{L}_a transformation.

The connection to the binomial transform of a slightly modified version $L_{a,b}$ (see Definition 4.1) of the L_a transformation is studied in Section 4. Theorem 4.2 gives a result that significantly generalizes a previously known special case described in Remark 4.4.

In Section 5 we will apply the inverse transformation of L_a to deduce new relations for many classical hypergeometric orthogonal polynomials. In particular, we derive relations for the Wilson, Racah, continuous Hahn, Hahn, and Jacobi polynomials as well as for the special cases of the Jacobi polynomials given by the Gegenbauer (or ultraspherical) polynomials, the Chebyshev polynomials of the first and second kind, and the Legendre (or spherical) polynomials. We also use Theorem 4.2 to demonstrate other relations for some of the orthogonal polynomials.

In Section 6 we will use the inverse transformations of L_a and \tilde{L}_a to derive new formulas for sums that involve terminating ${}_4F_3(1)$ hypergeometric series and sums that involve terminating ${}_5F_4(1)$ hypergeometric series. These new summation formulas are given in (6.1), (6.2), (6.3), and (6.4).

2 Hypergeometric series

Let p and q be nonnegative integers. Let $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, z \in \mathbb{C}$. The hypergeometric series of type ${}_pF_q$ that has numerator parameters a_1, a_2, \dots, a_p and denominator parameters b_1, b_2, \dots, b_q is defined by

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{n! (b_1)_n (b_2)_n \cdots (b_q)_n} z^n. \quad (2.1)$$

If no numerator parameter is a negative integer or zero, we need no denominator parameter to be a negative integer or zero. In this case, the series in (2.1) converges absolutely for all z if $p < q + 1$. If $p > q + 1$, the series converges only when $z = 0$. In the case $p = q + 1$, the series converges absolutely if $|z| < 1$ or if $|z| = 1$ and $\operatorname{Re}(\sum_{i=1}^q b_i - \sum_{i=1}^{q+1} a_i) > 0$ (see [1, p. 8]).

If a numerator parameter is a negative integer or zero, then, letting $-n$ be the nonpositive integer numerator parameter of least absolute value, only

the first $n + 1$ terms of the series (2.1) are nonzero and the series is said to terminate. In this case, we require that no denominator parameter be in the set $\{-n + 1, -n + 2, \dots, 0\}$. We note that (2.1) reduces to a polynomial in z of degree n .

When $z = 1$, we say that the series is of unit argument and of type ${}_pF_q(1)$. If $\sum_{i=1}^q b_i - \sum_{i=1}^p a_i = 1$, the series is called Saalschützian. In the case $p = q + 1$, if $1 + a_1 = b_1 + a_2 = \dots = b_q + a_{q+1}$, the series is called well-poised. A well-poised series that satisfies $a_2 = 1 + \frac{1}{2}a_1$ is called very-well-poised.

In deriving the inverse transformation of L_a , we will make use of a version of Dixon's theorem (see [1, p. 13]) that gives the sum of a terminating well-poised ${}_3F_2(1)$ series:

$${}_3F_2 \left(\begin{matrix} a, b, -n \\ 1 + a - b, 1 + a + n \end{matrix} \middle| 1 \right) = \frac{(1 + a)_n (1 + \frac{a}{2} - b)_n}{(1 + \frac{a}{2})_n (1 + a - b)_n}. \quad (2.2)$$

We will also use the Chu-Vandermonde formula (see [1, p. 3]) for the sum of a terminating ${}_2F_1(1)$ series:

$${}_2F_1 \left(\begin{matrix} -n, a \\ b \end{matrix} \middle| 1 \right) = \frac{(b - a)_n}{(b)_n}. \quad (2.3)$$

A terminating hypergeometric series with a nonpositive integer numerator parameter of least absolute value equal to $-n$ can be considered as a transformation of the form (1.1). As an example, certain terminating hypergeometric series can be considered as binomial transforms. Indeed, Eq. (2.3) can be written as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(a)_k}{(b)_k} = \frac{(b - a)_n}{(b)_n},$$

and therefore we can conclude that the binomial transform of the sequence $\left(\frac{(a)_n}{(b)_n} \right)_{n=0}^{\infty}$ is the sequence $\left(\frac{(b-a)_n}{(b)_n} \right)_{n=0}^{\infty}$.

3 The inverse of the L_a transformation

In this section we first prove a formula for the inverse of the L_a transformation. The inverse formula will be used in Sections 5 and 6 to obtain relations among classical orthogonal polynomials as well as formulas for sums that involve terminating hypergeometric series.

Theorem 3.1. Let $x = (x_n)_{n=0}^{\infty}$ be a sequence of complex numbers.

(a) For $a \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$,

$$L_a^{-1}(x)_n = \frac{1}{n!(1+a)_n} \sum_{k=0}^n \frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} x_k, \quad n \geq 0. \quad (3.1)$$

(b) When $a = 0$,

$$L_0^{-1}(x)_n = \frac{1}{(n!)^2} \left(-x_0 + 2 \sum_{k=0}^n \frac{(-n)_k}{(1+n)_k} x_k \right), \quad n \geq 0. \quad (3.2)$$

Proof. (a) When $a \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$, we have

$$\begin{aligned} & \frac{1}{n!(1+a)_n} \sum_{k=0}^n \frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} L_a(x)_k \\ &= \frac{1}{n!(1+a)_n} \sum_{k=0}^n \left(\frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} \sum_{m=0}^k (-k)_m (k+a)_m x_m \right) \\ &= \frac{1}{n!(1+a)_n} \sum_{m=0}^n \sum_{k=m}^n \frac{(-1)^m (a)_{k+m} \left(1 + \frac{a}{2}\right)_k (-n)_k}{(k-m)! \left(\frac{a}{2}\right)_k (1+a+n)_k} x_m \\ &= \frac{1}{n!(1+a)_n} \sum_{m=0}^n \sum_{t=0}^{n-m} \frac{(-1)^m (a)_{2m+t} \left(1 + \frac{a}{2}\right)_{m+t} (-n)_{m+t}}{t! \left(\frac{a}{2}\right)_{m+t} (1+a+n)_{m+t}} x_m \\ &= \frac{1}{n!(1+a)_n} \sum_{m=0}^n \left(\frac{(-1)^m (a)_{2m} \left(1 + \frac{a}{2}\right)_m (-n)_m}{\left(\frac{a}{2}\right)_m (1+a+n)_m} \right. \\ & \quad \left. \times {}_3F_2 \left(\begin{matrix} a+2m, 1 + \frac{a}{2} + m, -(n-m) \\ \frac{a}{2} + m, 1+a+n+m \end{matrix} \middle| 1 \right) x_m \right). \end{aligned}$$

By Eq. (2.2),

$${}_3F_2 \left(\begin{matrix} a+2m, 1 + \frac{a}{2} + m, -(n-m) \\ \frac{a}{2} + m, 1+a+n+m \end{matrix} \middle| 1 \right) = \begin{cases} 0, & 0 \leq m < n, \\ 1, & m = n. \end{cases}$$

Hence

$$\begin{aligned}
& \frac{1}{n!(1+a)_n} \sum_{k=0}^n \frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} L_a(x)_k \\
&= \frac{(-1)^n (a)_{2n} \left(1 + \frac{a}{2}\right)_n (-n)_n}{n!(1+a)_n \left(\frac{a}{2}\right)_n (1+a+n)_n} x_n \\
&= \frac{(a)_{2n} \left(1 + \frac{a}{2}\right)_n}{(1+a)_{2n} \left(\frac{a}{2}\right)_n} x_n \\
&= x_n,
\end{aligned}$$

which proves part (a).

- (b) Let $a = 0$. Taking the limit as $a \rightarrow 0$ on the right-hand side of (3.1), we obtain, for $n \geq 0$,

$$\begin{aligned}
L_0^{-1}(x)_n &= \lim_{a \rightarrow 0} \frac{1}{n!(1+a)_n} \sum_{k=0}^n \frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} x_k \\
&= \lim_{a \rightarrow 0} \frac{1}{n!(1+a)_n} \left(x_0 + \sum_{k=1}^n \frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} x_k \right) \\
&= \lim_{a \rightarrow 0} \frac{1}{n!(1+a)_n} \left(x_0 + 2 \sum_{k=1}^n \frac{(a+1)_{k-1} \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(1 + \frac{a}{2}\right)_{k-1} (1+a+n)_k} x_k \right) \\
&= \frac{1}{(n!)^2} \left(x_0 + 2 \sum_{k=1}^n \frac{(k-1)! k! (-n)_k}{k! (k-1)! (1+n)_k} x_k \right) \\
&= \frac{1}{(n!)^2} \left(x_0 + 2 \sum_{k=1}^n \frac{(-n)_k}{(1+n)_k} x_k \right) \\
&= \frac{1}{(n!)^2} \left(-x_0 + 2 \sum_{k=0}^n \frac{(-n)_k}{(1+n)_k} x_k \right).
\end{aligned}$$

□

Considering Theorem 3.1, we define the following related transformation to L_a :

Definition 3.2. Let $a \in \mathbb{C} \setminus \{-2, -3, -4, \dots\}$. We define the transformation

$$\begin{aligned}\tilde{L}_a : \omega &\rightarrow \omega \\ x = (x_n)_{n=0}^\infty &\mapsto \tilde{L}_a(x) = (\tilde{L}_a(x)_n)_{n=0}^\infty\end{aligned}$$

by

$$\tilde{L}_a(x)_n = \sum_{k=0}^n \frac{(-n)_k}{(1+a+n)_k} x_k, \quad n \geq 0, \quad (3.3)$$

We need the restriction $a \notin \{-2, -3, -4, \dots\}$ so that the \tilde{L}_a transformation is well-defined. The inverse of the \tilde{L}_a transformation is given in the next theorem:

Theorem 3.3. Let $x = (x_n)_{n=0}^\infty$ be a sequence of complex numbers.

(a) For $a \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$,

$$\tilde{L}_a^{-1}(x)_n = \frac{(a)_n (1 + \frac{a}{2})_n}{n! (\frac{a}{2})_n} \sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (1+a)_k} x_k, \quad n \geq 0. \quad (3.4)$$

(b) When $a = 0$,

$$\tilde{L}_0^{-1}(x)_n = \begin{cases} x_0, & n = 0, \\ 2 \sum_{k=0}^n \frac{(-n)_k (n)_k}{k! k!} x_k, & n > 0. \end{cases} \quad (3.5)$$

(c) When $a = -1$,

$$\tilde{L}_{-1}^{-1}(x)_n = \begin{cases} x_0, & n = 0, \\ (2n-1) \sum_{k=0}^n \frac{(-1)^k (k)_{n-1}}{(n-k)! k!} x_k, & n > 0. \end{cases} \quad (3.6)$$

Proof. Part (a) follows from (3.1) and Remark 1.1. Part (b) follows by taking the limit as $a \rightarrow 0$ in part (a) in a similar way as was done in the proof of part (b) of Theorem 3.1.

To prove part (c), let $a = -1$. From the definition of $\tilde{L}_{-1}(x)_n$, it is clear that $\tilde{L}_{-1}(x)_0 = x_0$. Now assume $n > 0$. Taking the limit as $a \rightarrow -1$ on the

right-hand side of (3.4), we obtain, for $n > 0$,

$$\begin{aligned}
\tilde{L}_{-1}^{-1}(x)_n &= \lim_{a \rightarrow -1} \frac{(a)_n \left(1 + \frac{a}{2}\right)_n}{n! \left(\frac{a}{2}\right)_n} \sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (1+a)_k} x_k \\
&= \lim_{a \rightarrow -1} \frac{2(1+a)_{n-1} \left(1 + \frac{a}{2}\right)_n}{n! \left(1 + \frac{a}{2}\right)_{n-1}} \left(\sum_{k=0}^{n-1} \frac{(-n)_k (n+a)_k}{k! (1+a)_k} x_k + \frac{(-n)_n (n+a)_n}{n! (1+a)_n} x_n \right) \\
&= \lim_{a \rightarrow -1} \frac{2(1+a)_{n-1} \left(n + \frac{a}{2}\right)}{n!} \left(\sum_{k=0}^{n-1} \frac{(-n)_k (n+a)_k}{k! (1+a)_k} x_k + \frac{(-n)_n (n+a)_n}{n! (1+a)_n} x_n \right) \\
&= \lim_{a \rightarrow -1} \frac{2n+a}{n!} \left(\sum_{k=0}^{n-1} \frac{(-n)_k (n+a)_k (1+a+k)_{n-1-k}}{k!} x_k \right. \\
&\quad \left. + \frac{(-n)_n (n+a)_n}{n! (n+a)} x_n \right) \\
&= \lim_{a \rightarrow -1} \frac{2n+a}{n!} \left(\sum_{k=0}^{n-1} \frac{(-n)_k (n+a)_k (1+a+k)_{n-1-k}}{k!} x_k \right. \\
&\quad \left. + \frac{(-n)_n (n+a+1)_{n-1}}{n!} x_n \right) \\
&= \frac{2n-1}{n!} \left(\sum_{k=0}^{n-1} \frac{(-n)_k (n-1)_k (k)_{n-1-k}}{k!} x_k + \frac{(-n)_n (n)_{n-1}}{n!} x_n \right) \\
&= \frac{2n-1}{n!} \left(\sum_{k=0}^{n-1} \frac{(-n)_k (k)_{n-1}}{k!} x_k + \frac{(-n)_n (n)_{n-1}}{n!} x_n \right) \\
&= (2n-1) \left(\sum_{k=0}^{n-1} \frac{(-1)^k (k)_{n-1}}{(n-k)! k!} x_k + \frac{(-1)^n (n)_{n-1}}{n!} x_n \right) \\
&= (2n-1) \sum_{k=0}^n \frac{(-1)^k (k)_{n-1}}{(n-k)! k!} x_k.
\end{aligned}$$

□

We remark that (3.6) can also be written as

$$\tilde{L}_{-1}^{-1}(x)_n = \begin{cases} x_0, & n = 0, \\ x_0 - x_1, & n = 1, \\ (1-2n) \sum_{k=0}^{n-1} \frac{(1-n)_k (n)_k}{k! (2)_k} x_{k+1}, & n > 1. \end{cases} \quad (3.7)$$

Indeed, when $n = 1$,

$$(2n - 1) \sum_{k=0}^n \frac{(-1)^k (k)_{n-1}}{(n-k)!k!} x_k = \sum_{k=0}^1 \frac{(-1)^k}{(1-k)!k!} x_k = x_0 - x_1.$$

When $n > 1$,

$$\begin{aligned} & (2n - 1) \sum_{k=0}^n \frac{(-1)^k (k)_{n-1}}{(n-k)!k!} x_k \\ &= (2n - 1) \sum_{k=1}^n \frac{(-1)^k (n+k-2)!}{(n-k)!(k-1)!k!} x_k \\ &= (2n - 1) \sum_{k=1}^n \frac{(-1)^k (n-k+1)_{2k-2}}{(k-1)!k!} x_k \\ &= (2n - 1) \sum_{k=1}^n \frac{(-1)^k (n-k+1)_{k-1} (n)_{k-1}}{(k-1)!k!} x_k \\ &= (2n - 1) \sum_{k=1}^n \frac{(-1)^k (-1)^{k-1} (1-n)_{k-1} (n)_{k-1}}{(k-1)!k!} x_k \\ &= (1 - 2n) \sum_{k=0}^{n-1} \frac{(1-n)_k (n)_k}{k!(k+1)!} x_{k+1} \\ &= (1 - 2n) \sum_{k=0}^{n-1} \frac{(1-n)_k (n)_k}{k!(2)_k} x_{k+1}. \end{aligned}$$

4 Connection to the binomial transform

In this section we explore a connection with the binomial transform. For this purpose, we need to modify slightly the definition of the L_a transformation and make the following definition:

Definition 4.1. Let $a, b \in \mathbb{C}$, $a \notin \{-1, -2, -3, \dots\}$, $b \notin \{0, -1, -2, -3, \dots\}$. We define the transformation

$$\begin{aligned} L_{a,b} : \omega &\rightarrow \omega \\ x = (x_n)_{n=0}^\infty &\mapsto L_{a,b}(x) = (L_{a,b}(x)_n)_{n=0}^\infty \end{aligned}$$

by

$$L_{a,b}(x)_n = \sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (b)_k} x_k, \quad n \geq 0. \quad (4.1)$$

We note that by Theorem 3.1, it follows that the inverse of the $L_{a,b}$ transformation is given by:

(a) For $a \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$,

$$L_{a,b}^{-1}(x)_n = \frac{(b)_n}{(1+a)_n} \sum_{k=0}^n \frac{(a)_k \left(1 + \frac{a}{2}\right)_k (-n)_k}{k! \left(\frac{a}{2}\right)_k (1+a+n)_k} x_k, \quad n \geq 0. \quad (4.2)$$

(b) When $a = 0$,

$$L_{0,b}^{-1}(x)_n = \frac{(b)_n}{n!} \left(-x_0 + 2 \sum_{k=0}^n \frac{(-n)_k}{(1+n)_k} x_k \right), \quad n \geq 0. \quad (4.3)$$

A connection between the $L_{a,b}$ transformation and the binomial transform is given in Theorem 4.2 below. The result in Theorem 4.2 significantly generalizes a previously known special case described in Remark 4.4.

Theorem 4.2. *Let $x = (x_n)_{n=0}^\infty$ be a sequence of complex numbers and let $\hat{x} = (\hat{x}_n)_{n=0}^\infty$ be its binomial transform. Then if $b \notin \{1+a, 2+a, 3+a, \dots\}$, we have*

$$L_{a,b}(\hat{x})_n = (-1)^n \frac{(1+a-b)_n}{(b)_n} L_{a,1+a-b}(x)_n. \quad (4.4)$$

Proof. We have

$$\begin{aligned}
& L_{a,b}(\hat{x})_n \\
&= \sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (b)_k} \hat{x}_k \\
&= \sum_{k=0}^n \left(\frac{(-n)_k (n+a)_k}{k! (b)_k} \sum_{m=0}^k (-1)^m \binom{k}{m} x_m \right) \\
&= \sum_{m=0}^n \sum_{k=m}^n \frac{(-1)^m (-n)_k (n+a)_k}{(b)_k m! (k-m)!} x_m \\
&= \sum_{m=0}^n \sum_{t=0}^{n-m} \frac{(-1)^m (-n)_{m+t} (n+a)_{m+t}}{(b)_{m+t} m! t!} x_m \\
&= \sum_{m=0}^n \left(\frac{(-1)^m (-n)_m (n+a)_m}{(b)_m m!} \right. \\
&\quad \left. \times {}_2F_1 \left(\begin{matrix} -(n-m), n+a+m \\ b+m \end{matrix} \middle| 1 \right) x_m \right).
\end{aligned}$$

By Eq. (2.3),

$$\begin{aligned}
& {}_2F_1 \left(\begin{matrix} -(n-m), n+a+m \\ b+m \end{matrix} \middle| 1 \right) \\
&= \frac{(b-a-n)_{n-m}}{(b+m)_{n-m}} \\
&= \frac{(b-a-n)_{n-m}}{(-1)^{n-m} (1-b-n)_{n-m}} \\
&= (-1)^{n+m} \frac{(1+a-b)_n (b)_m}{(1+a-b)_m (b)_n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& L_{a,b}(\hat{x})_n \\
&= (-1)^n \frac{(1+a-b)_n}{(b)_n} \sum_{m=0}^n \frac{(-n)_m (n+a)_m}{m! (1+a-b)_m} x_m \\
&= (-1)^n \frac{(1+a-b)_n}{(b)_n} L_{a,1+a-b}(x)_n.
\end{aligned}$$

□

We note that (4.4) can also be written as

$$\sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (b)_k} \hat{x}_k = (-1)^n \frac{(1+a-b)_n}{(b)_n} \sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (1+a-b)_k} x_k. \quad (4.5)$$

Remark 4.3. Some authors define the binomial transform of a sequence $(x_n)_{n=0}^\infty$ by

$$y_n = \sum_{k=0}^n \binom{n}{k} x_k, \quad n \geq 0. \quad (4.6)$$

With the definition from above, Equation (4.5) becomes

$$\sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! (b)_k} y_k = (-1)^n \frac{(1+a-b)_n}{(b)_n} \sum_{k=0}^n (-1)^k \frac{(-n)_k (n+a)_k}{k! (1+a-b)_k} x_k. \quad (4.7)$$

It is interesting to note that the special case $b = \frac{1+a}{2}$ in (4.4) yields

$$L_{a, \frac{1+a}{2}}(\hat{x})_n = (-1)^n L_{a, \frac{1+a}{2}}(x)_n, \quad (4.8)$$

i.e.

$$\sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! \left(\frac{1+a}{2}\right)_k} \hat{x}_k = (-1)^n \sum_{k=0}^n \frac{(-n)_k (n+a)_k}{k! \left(\frac{1+a}{2}\right)_k} x_k. \quad (4.9)$$

When a is a positive odd integer in (4.8), i.e. when $a = 2r + 1, r \in \{0, 1, 2, \dots\}$, using

$$\frac{(n+2r+1)_k}{(r+1)_k} = \binom{n+k+2r}{k+r} \frac{r!}{(n+r+1)_r},$$

we obtain from (4.9) that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k+2r}{k+r} \hat{x}_k = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k+2r}{k+r} x_k. \quad (4.10)$$

Remark 4.4. Recently, Sun [7, Theorem 2.1] proved the special case of (4.10) when $r = 0$ and the binomial transform is defined as in (4.6). Using generalized Seidel matrices, Chen [2, Theorem 3.2] proved a slightly different form of (4.10), again with the binomial transform as defined in (4.6). If we take $m = n = s$ in [2, Theorem 3.2], we obtain the special case of (4.10) when $r = 0$ that is also proven in [7, Theorem 2.1].

5 Connection to classical hypergeometric orthogonal polynomials

In this section we show the connection of the inverse of the L_a transformation to classical hypergeometric orthogonal polynomials. In particular, we use Theorem 3.1 to obtain new relations for the Wilson, Racah, continuous Hahn, Hahn, and Jacobi polynomials. We also give the corresponding relations for the special cases of the Jacobi polynomials given by the Gegenbauer (or ultraspherical) polynomials, the Chebyshev polynomials of the first and second kind, and the Legendre (or spherical) polynomials. Using Theorem 4.2, we demonstrate further relations for some of the orthogonal polynomials. For each type of orthogonal polynomials, we start with its definition as given in [4] and then state our relations.

5.1 Wilson polynomials

The Wilson polynomials $W_n(x^2; a, b, c, d)$ are defined by

$$\begin{aligned} & W_n(x^2; a, b, c, d) \\ &= (a+b)_n(a+c)_n(a+d)_n \\ & \times {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} \middle| 1 \right). \end{aligned}$$

By (3.1), if $a+b+c+d \notin \{1, 0, -1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \left(\frac{(a+b+c+d-1)_k \left(\frac{a+b+c+d+1}{2} \right)_k (-n)_k}{k! \left(\frac{a+b+c+d-1}{2} \right)_k (a+b+c+d+n)_k (a+b)_k (a+c)_k (a+d)_k} \right. \\ & \times W_k(x^2; a, b, c, d) \Big) \\ &= \frac{(a+b+c+d)_n (a+ix)_n (a-ix)_n}{(a+b)_n (a+c)_n (a+d)_n}, \quad n \geq 0. \end{aligned} \quad (5.1)$$

When $a+b+c+d = 1$, using (3.2) and the fact that $W_0(x^2; a, b, c, d) = 1$, we obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{(-n)_k}{(1+n)_k (a+b)_k (a+c)_k (a+d)_k} W_k(x^2; a, b, c, d) \\ &= \frac{n! (a+ix)_n (a-ix)_n + (a+b)_n (a+c)_n (a+d)_n}{2(a+b)_n (a+c)_n (a+d)_n}, \quad n \geq 0. \end{aligned} \quad (5.2)$$

5.2 Racah polynomials

The Racah polynomials are defined by

$$\begin{aligned} & R_n(\lambda(x); \alpha, \beta, \gamma, \delta) \\ &= {}_4F_3 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix} \middle| 1 \right), \\ & n = 0, 1, 2, \dots, N, \end{aligned}$$

where

$$\lambda(x) = x(x + \gamma + \delta + 1)$$

and

$\alpha + 1 = -N$ or $\beta + \delta + 1 = -N$ or $\gamma + 1 = -N$, with N a non-negative integer.

By (3.1), if $\alpha + \beta \notin \{-1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \frac{(\alpha + \beta + 1)_k \left(\frac{\alpha + \beta + 3}{2}\right)_k (-n)_k}{k! \left(\frac{\alpha + \beta + 1}{2}\right)_k (2 + \alpha + \beta + n)_k} R_k(\lambda(x); \alpha, \beta, \gamma, \delta) \\ &= \frac{(\alpha + \beta + 2)_n (-x)_n (x + \gamma + \delta + 1)_n}{(\alpha + 1)_n (\beta + \delta + 1)_n (\gamma + 1)_n}, \quad n \geq 0. \end{aligned} \quad (5.3)$$

When $\alpha + \beta = -1$, using (3.2) and the fact that $R_0(\lambda(x); \alpha, \beta, \gamma, \delta) = 1$, we obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{(-n)_k}{(1 + n)_k} R_k(\lambda(x); \alpha, \beta, \gamma, \delta) \\ &= \frac{n! (-x)_n (x + \gamma + \delta + 1)_n + (\alpha + 1)_n (\beta + \delta + 1)_n (\gamma + 1)_n}{2(\alpha + 1)_n (\beta + \delta + 1)_n (\gamma + 1)_n}, \quad n \geq 0. \end{aligned} \quad (5.4)$$

5.3 Continuous Hahn polynomials

The continuous Hahn polynomials $p_n(x; a, b, c, d)$ are defined by

$$\begin{aligned} & p_n(x; a, b, c, d) \\ &= i^n \frac{(a + c)_n (a + d)_n}{n!} \\ & \times {}_3F_2 \left(\begin{matrix} -n, n + a + b + c + d - 1, a + ix \\ a + c, a + d \end{matrix} \middle| 1 \right). \end{aligned}$$

By (3.1), if $a + b + c + d \notin \{1, 0, -1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \left((-i)^k \frac{(a+b+c+d-1)_k \left(\frac{a+b+c+d+1}{2} \right)_k (-n)_k}{\left(\frac{a+b+c+d-1}{2} \right)_k (a+b+c+d+n)_k (a+c)_k (a+d)_k} \right. \\ & \quad \times p_k(x; a, b, c, d) \Big) \\ &= \frac{(a+b+c+d)_n (a+ix)_n}{(a+c)_n (a+d)_n}, \quad n \geq 0. \end{aligned} \quad (5.5)$$

When $a + b + c + d = 1$, using (3.2) and the fact that $p_0(x; a, b, c, d) = 1$, we obtain

$$\begin{aligned} & \sum_{k=0}^n (-i)^k \frac{(-n)_k k!}{(1+n)_k (a+c)_k (a+d)_k} p_k(x; a, b, c, d) \\ &= \frac{n! (a+ix)_n + (a+c)_n (a+d)_n}{2(a+c)_n (a+d)_n}, \quad n \geq 0. \end{aligned} \quad (5.6)$$

By (2.3), the binomial transform of the sequence $\left(\frac{(d-ix)_n}{(a+d)_n} \right)_{n=0}^{\infty}$ is the sequence $\left(\frac{(a+ix)_n}{(a+d)_n} \right)_{n=0}^{\infty}$. Using this along with (4.4), we obtain

$$\begin{aligned} & p_n(x; a, b, c, d) \\ &= i^n \frac{(a+c)_n (a+d)_n}{n!} \\ & \quad \times {}_3F_2 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} \middle| 1 \right) \\ &= (-1)^n i^n \frac{(a+d)_n (b+d)_n}{n!} \\ & \quad \times {}_3F_2 \left(\begin{matrix} -n, n+a+b+c+d-1, d-ix \\ b+d, a+d \end{matrix} \middle| 1 \right) \\ &= (-1)^n p_n(-x; d, c, b, a). \end{aligned} \quad (5.7)$$

Combining the above with the trivial invariances of $p_n(x; a, b, c, d)$ under interchanging c and d , we obtain eight identities of the form

$$p_n(x; a, b, c, d) = (-1)^{kn} p_n((-1)^k x; x_1, x_2, x_3, x_4), \quad (5.8)$$

where $k \in \{0, 1\}$ and (x_1, x_2, x_3, x_4) is a permutation of (a, b, c, d) such that $x_1, x_2 \in \{a, b\}, x_3, x_4 \in \{c, d\}$ if $k = 0$, and $x_1, x_2 \in \{c, d\}, x_3, x_4 \in \{a, b\}$ if $k = 1$.

5.4 Hahn polynomials

The Hahn polynomials $Q_n(x; \alpha, \beta, N)$ are defined by

$$\begin{aligned} & Q_n(x; \alpha, \beta, N) \\ &= {}_3F_2 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{matrix} \middle| 1 \right), \\ & n = 0, 1, 2, \dots, N. \end{aligned}$$

By (3.1), if $\alpha + \beta \notin \{-1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \frac{(\alpha + \beta + 1)_k \left(\frac{\alpha + \beta + 3}{2}\right)_k (-n)_k}{k! \left(\frac{\alpha + \beta + 1}{2}\right)_k (2 + \alpha + \beta + n)_k} Q_k(x; \alpha, \beta, N) \\ &= \frac{(\alpha + \beta + 2)_n (-x)_n}{(\alpha + 1)_n (-N)_n}, \quad n \geq 0. \end{aligned} \quad (5.9)$$

When $\alpha + \beta = -1$, i.e. when $\beta = -\alpha - 1$, using (3.2) and the fact that $Q_0(x; \alpha, \beta, N) = 1$, we obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{(-n)_k}{(1 + n)_k} Q_k(x; \alpha, -\alpha - 1, N) \\ &= \frac{n!(-x)_n + (\alpha + 1)_n (-N)_n}{2(\alpha + 1)_n (-N)_n}, \quad n \geq 0. \end{aligned} \quad (5.10)$$

By (2.3), the binomial transform of the sequence $\left(\frac{(-N+x)_n}{(-N)_n}\right)_{n=0}^{\infty}$ is the sequence $\left(\frac{(-x)_n}{(-N)_n}\right)_{n=0}^{\infty}$. Using this along with (4.4), we obtain

$$\begin{aligned} & Q_n(x; \alpha, \beta, N) \\ &= {}_3F_2 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{matrix} \middle| 1 \right) \\ &= (-1)^n \frac{(\beta + 1)_n}{(\alpha + 1)_n} {}_3F_2 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -N + x \\ \beta + 1, -N \end{matrix} \middle| 1 \right) \\ &= (-1)^n \frac{(\beta + 1)_n}{(\alpha + 1)_n} Q_n(N - x; \beta, \alpha, N). \end{aligned} \quad (5.11)$$

5.5 Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ are defined by

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} \middle| \frac{1-x}{2} \right). \end{aligned}$$

By (3.1), if $\alpha + \beta \notin \{-1, -2, -3, \dots\}$,

$$\begin{aligned} &\sum_{k=0}^n \frac{(\alpha + \beta + 1)_k \left(\frac{\alpha + \beta + 3}{2}\right)_k (-n)_k}{\left(\frac{\alpha + \beta + 1}{2}\right)_k (2 + \alpha + \beta + n)_k (\alpha + 1)_k} P_k^{(\alpha, \beta)}(x) \\ &= \frac{(\alpha + \beta + 2)_n}{(\alpha + 1)_n} \left(\frac{1-x}{2}\right)^n, \quad n \geq 0. \end{aligned} \quad (5.12)$$

When $\alpha + \beta = -1$, i.e. when $\beta = -\alpha - 1$, using (3.2) and the fact that $P_0^{(\alpha, \beta)}(x) = 1$, we obtain

$$\begin{aligned} &\sum_{k=0}^n \frac{(-n)_k k!}{(1+n)_k (\alpha + 1)_k} P_k^{(\alpha, -\alpha-1)}(x) \\ &= \frac{n! \left(\frac{1-x}{2}\right)^n + (\alpha + 1)_n}{2(\alpha + 1)_n}, \quad n \geq 0. \end{aligned} \quad (5.13)$$

By the binomial theorem, the binomial transform of the sequence $\left(\frac{1+x}{2}\right)_{n=0}^\infty$ is the sequence $\left(\frac{1-x}{2}\right)_{n=0}^\infty$. Using this along with (4.4), we obtain

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} \middle| \frac{1-x}{2} \right) \\ &= (-1)^n \frac{(\beta + 1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \beta + 1 \end{matrix} \middle| \frac{1+x}{2} \right) \\ &= (-1)^n P_n^{(\beta, \alpha)}(-x). \end{aligned}$$

The above formula for the Jacobi polynomials is well-known (see [8, p. 58]).

There are also several special cases of the Jacobi polynomials that we consider below.

5.5.1 Gegenbauer (or ultraspherical) polynomials

The Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(x)$ are defined by

$$\begin{aligned} C_n^{(\lambda)}(x) &= \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x) \\ &= \frac{(2\lambda)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n + 2\lambda \\ \lambda + \frac{1}{2} \end{matrix} \middle| \frac{1-x}{2} \right), \quad \lambda \neq 0. \end{aligned}$$

By (3.1),

$$\begin{aligned} &\sum_{k=0}^n \frac{(1+\lambda)_k (-n)_k}{(\lambda)_k (1+2\lambda+n)_k} C_k^{(\lambda)}(x) \\ &= \frac{(1+2\lambda)_n}{(\lambda + \frac{1}{2})_n} \left(\frac{1-x}{2} \right)^n, \quad n \geq 0. \end{aligned} \tag{5.14}$$

5.5.2 Chebyshev polynomials of the first kind

The Chebyshev polynomials of the first kind $T_n(x)$ are defined by

$$\begin{aligned} T_n(x) &= \frac{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)}{P_n^{(-\frac{1}{2}, -\frac{1}{2})}(1)} \\ &= {}_2F_1 \left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} \middle| \frac{1-x}{2} \right). \end{aligned}$$

Using (3.2) and the fact that $T_0(x) = 1$, we obtain,

$$\begin{aligned} &\sum_{k=0}^n \frac{(-n)_k}{(1+n)_k} T_k(x) \\ &= \frac{n! \left(\frac{1-x}{2} \right)^n + \left(\frac{1}{2} \right)_n}{2 \left(\frac{1}{2} \right)_n}, \quad n \geq 0. \end{aligned} \tag{5.15}$$

5.5.3 Chebyshev polynomials of the second kind

The Chebyshev polynomials of the second kind $U_n(x)$ are defined by

$$\begin{aligned} U_n(x) &= (n+1) \frac{P_n^{(\frac{1}{2}, \frac{1}{2})}(x)}{P_n^{(\frac{1}{2}, \frac{1}{2})}(1)} \\ &= (n+1) {}_2F_1 \left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix} \middle| \frac{1-x}{2} \right). \end{aligned}$$

By (3.1),

$$\begin{aligned} &\sum_{k=0}^n \frac{(2)_k (-n)_k}{k! (2+n)_k} U_k(x) \\ &= \frac{(3)_n}{(\frac{3}{2})_n} \left(\frac{1-x}{2} \right)^n, \quad n \geq 0. \end{aligned} \tag{5.16}$$

5.5.4 Legendre (or spherical) polynomials

The Legendre (or spherical) polynomials $P_n(x)$ are defined by

$$\begin{aligned} P_n(x) &= P_n^{(0,0)}(x) \\ &= {}_2F_1 \left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2} \right). \end{aligned}$$

By (3.1),

$$\begin{aligned} &\sum_{k=0}^n \frac{(\frac{3}{2})_k (-n)_k}{(\frac{1}{2})_k (2+n)_k} P_k(x) \\ &= (n+1) \left(\frac{1-x}{2} \right)^n, \quad n \geq 0. \end{aligned} \tag{5.17}$$

6 Connection to sums that involve terminating hypergeometric series

In this section we explore how Theorems 3.1 and 3.3 lead to new formulas for sums that involve terminating ${}_4F_3(1)$ hypergeometric series and sums that involve terminating ${}_5F_4(1)$ hypergeometric series. These new summation formulas are given in (6.1), (6.2), (6.3), and (6.4) below.

6.1 Summations involving ${}_4F_3(1)$ series

From [1, Eq. 4.3.4],

$$\begin{aligned} & {}_7F_6 \left(\begin{matrix} a, 1 + \frac{a}{2}, b, c, d, e, -n \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + n \end{matrix} \middle| 1 \right) \\ &= \frac{(1+a)_n(1+a-d-e)_n}{(1+a-d)_n(1+a-e)_n} {}_4F_3 \left(\begin{matrix} 1+a-b-c, d, e, -n \\ 1+a-b, 1+a-c, d+e-a-n \end{matrix} \middle| 1 \right). \end{aligned}$$

Hence by (3.4), if $a \notin \{0, -1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \left(\frac{(-n)_k(n+a)_k(1+a-d-e)_k}{k!(1+a-d)_k(1+a-e)_k} \right. \\ & \quad \times {}_4F_3 \left(\begin{matrix} 1+a-b-c, d, e, -k \\ 1+a-b, 1+a-c, d+e-a-k \end{matrix} \middle| 1 \right) \Bigg) \\ &= \frac{(b)_n(c)_n(d)_n(e)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-e)_n}. \end{aligned} \quad (6.1)$$

Using $n+a$ in place of b and b in place of a in [5, Eq. 7.6.2.15], we have

$$\begin{aligned} & {}_5F_4 \left(\begin{matrix} -n, n+a, b, b + \frac{1}{2}, c \\ \frac{a}{2}, \frac{1+a}{2}, d, 1+2b+c-d \end{matrix} \middle| 1 \right) \\ &= \frac{(a-2b)_n}{(a)_n} {}_4F_3 \left(\begin{matrix} -n, 2b, d-c, 1+2b-d \\ 1+2b-a-n, d, 1+2b+c-d \end{matrix} \middle| 1 \right). \end{aligned}$$

Hence by (3.1), if $a \notin \{0, -1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \left(\frac{(-n)_k \left(1 + \frac{a}{2}\right)_k (a-2b)_k}{k!(1+a+n)_k \left(\frac{a}{2}\right)_k} \right. \\ & \quad \times {}_4F_3 \left(\begin{matrix} -k, 2b, d-c, 1+2b-d \\ 1+2b-a-k, d, 1+2b+c-d \end{matrix} \middle| 1 \right) \Bigg) \\ &= \frac{(1+a)_n(b)_n \left(b + \frac{1}{2}\right)_n (c)_n}{\left(\frac{a}{2}\right)_n \left(\frac{1+a}{2}\right)_n (d)_n (1+2b+c-d)_n}. \end{aligned} \quad (6.2)$$

6.2 Summations involving ${}_5F_4(1)$ series

From [1, Eq. 4.6.1],

$$\begin{aligned} & {}_4F_3 \left(\begin{matrix} b, c, d, -n \\ 1+a-c, 1+a-d, 1+a+n \end{matrix} \middle| 1 \right) \\ &= \frac{(1+a)_n(1+a-c-d)_n}{(1+a-c)_n(1+a-d)_n} \\ & \times {}_5F_4 \left(\begin{matrix} 1+\frac{a-b}{2}, \frac{1+a-b}{2}, c, d, -n \\ 1+\frac{a}{2}, 1+a-b, \frac{1+a}{2}, c+d-a-n \end{matrix} \middle| 1 \right). \end{aligned}$$

Hence by (3.4), if $a \notin \{0, -1, -2, -3, \dots\}$,

$$\begin{aligned} & \sum_{k=0}^n \left(\frac{(-n)_k(n+a)_k(1+a-c-d)_k}{k!(1+a-c)_k(1+a-d)_k} \right. \\ & \times {}_5F_4 \left(\begin{matrix} 1+\frac{a-b}{2}, \frac{1+a-b}{2}, c, d, -k \\ 1+\frac{a}{2}, 1+a-b, \frac{1+a}{2}, c+d-a-k \end{matrix} \middle| 1 \right) \Bigg) \\ &= \frac{(b)_n \left(\frac{a}{2}\right)_n (c)_n (d)_n}{(a)_n \left(1+\frac{a}{2}\right)_n (1+a-c)_n (1+a-d)_n}. \end{aligned} \tag{6.3}$$

From [1, Eq. 4.6.2],

$$\begin{aligned} & {}_5F_4 \left(\begin{matrix} b, 1+\frac{a}{2}, c, d, -n \\ \frac{a}{2}, 1+a-c, 1+a-d, 1+a+n \end{matrix} \middle| 1 \right) \\ &= \frac{(1+a)_n(1+a-c-d)_n}{(1+a-c)_n(1+a-d)_n} \\ & \times {}_5F_4 \left(\begin{matrix} \frac{a-b}{2}, \frac{1+a-b}{2}, c, d, -n \\ \frac{a}{2}, 1+a-b, \frac{1+a}{2}, c+d-a-n \end{matrix} \middle| 1 \right). \end{aligned}$$

Hence by (3.4), if $a \notin \{0, -1, -2, -3, \dots\}$,

$$\begin{aligned}
& \sum_{k=0}^n \left(\frac{(-n)_k (n+a)_k (1+a-c-d)_k}{k! (1+a-c)_k (1+a-d)_k} \right. \\
& \quad \times {}_5F_4 \left(\begin{matrix} \frac{a-b}{2}, \frac{1+a-b}{2}, c, d, -k \\ \frac{a}{2}, 1+a-b, \frac{1+a}{2}, c+d-a-k \end{matrix} \middle| 1 \right) \Bigg) \\
& = \frac{(b)_n (c)_n (d)_n}{(a)_n (1+a-c)_n (1+a-d)_n}.
\end{aligned} \tag{6.4}$$

References

- [1] W.N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, Cambridge, 1935.
- [2] K.-W. Chen, Identities from the binomial transform, J. Number Theory, **124** (2007), no. 1, 142–150.
- [3] D.E. Knuth, The Art of Computer Programming, vol. 3: Sorting and Searching, Addison-Wesley, 1973.
- [4] R. Koekoek, R. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue, Reports of the Faculty of Technical Mathematics and Informatics, No. 98-17, Delft University of Technology, Delft, 1998.
- [5] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, Vol. 3, More Special Functions, Translated from the Russian by G.G. Gould, Gordon and Breach Science Publishers, New York, 1990.
- [6] J. Riordan, Combinatorial Identities, John Wiley & Sons, New York, 1968.
- [7] Z.-W. Sun, Congruences involving $g_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} x^k$, Ramanujan J., 2015, DOI 10.1007/s11139-015-9727-3.
- [8] G. Szegő, Orthogonal Polynomials, American Mathematical Society, New York City, 1939.